Packing Segments in a Convex 3-Polytope is NP-hard

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Abstract

We show it is NP-hard to pack the maximum number of segments in a convex 3-dimensional polytope. We show this packing problem is also NP-hard for general polygonal regions in the plane. This problem relates two streams of research, Kakeya set problems and packing problems.

1 Introduction

1.1 The Kakeya Set Problem

A translation cover of a collection $\mathcal{C}$ of geometric objects is a region that contains some translate of each object in $\mathcal{C}$. Julius Pál showed that an equilateral triangle of area $1/\sqrt{3}$ is a minimum-area convex translation cover for all unit-length segments in the plane [8].

A variant of this question had been posed by Soichi Kakeya with the additional requirement that a unit-length segment can be continuously rotated by $[0, \pi]$ inside the region and return to its initial position [5]. This problem was also resolved by Pál [7].

Many variations of this problem have been considered, and these have lead to some important applications in analysis. Terence Tao has given a nice self-contained survey of several of these applications [10], and more recently Izabella Łaba has given a more extensive survey [6].

A recent variation of these problems is to consider arbitrary collections of segments instead of unit-length segments. This has lead to an $O(n \log n)$ time algorithm that constructs a minimum-area translation cover of a given collection of n segments [1].

1.2 Motivation and Results

The following minimum-packaging problem is a natural extension of [1]. Given a collection of segments, what is the minimum-area convex set containing a disjoint translate of each segment? In contrast, we suspect that this problem is intractable.

Our understanding of this minimum-packaging problem may benefit from an understanding of the following analogous maximum-packing problem. Given both a collection of segments and a region in $\mathbb{R}^d$, what is the maximum number of segments that can be disjointly embedded in the region by translation? We call this the maximum segment-packing problem.

In this paper, we prove that the maximum segment-packing problem is NP-hard in the following cases.

(I) When the region is restricted to being a convex 3-dimensional polytope.

(II) When the region is a general region in the plane.

The proof of both cases will use similar arguments. The complexity of the problem for convex polygonal regions in the plane remains open.

2 Results

2.1 Problem Definition

MaxSegPack$d$ for a class $\mathcal{R}$ of regions in $\mathbb{R}^d$ denotes the following problem. Given a collection of segments $\mathcal{S}$ and a region $R \in \mathcal{R}$, what is the maximum number of segments that can be disjointly embedded in R?

2.2 Case I

Theorem 1 *MaxSegPack3 for convex polytopes is NP-hard.*

We proceed by reduction to a known NP-hard problem. It is well-known that finding the maximum independent set of vertices of a graph (MaxIndSet) is NP-hard. Moreover, MaxIndSet restricted to a bridgeless triangle-free cubic graph is still NP-hard [11]. Given such a graph G, we will construct an instance of MaxSegPack3 for convex polytopes. For this we will use the following lemma.

Lemma 2 *If G is a bridgeless triangle-free cubic graph, then there exists a collection of lines $\mathcal{L}$ in $\mathbb{R}^3$ such that G is the incidence graph of $\mathcal{L}$.*

2.3 Construction of lines $\mathcal{L}$

We denote the set of vertices in G as $V(G)$ and the set of edges in G as $E(G)$. Let G be a bridgeless triangle-free cubic graph. We will construct a collection $\mathcal{L}$ consisting of a line ℓ_v for each vertex $v \in V(G)$.

This can be done in the following way. First, since G is bridgeless and cubic, it has a perfect matching $M \subset E(G)$ by Petersen's theorem [9]. Now generate

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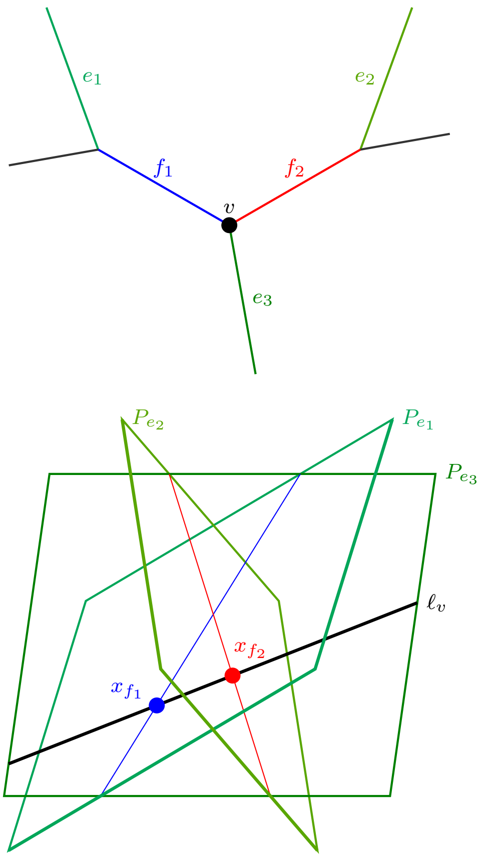

Figure 1: Contruction of $\mathcal{P}$, $\mathcal{X}$, and $\mathcal{L}$ from G.

a collection of planes $\mathcal{P}$ consisting of a plane P_e for each edge of the matching $e \in M$, and then generate a collection of points $\mathcal{X}$ consisting of a point x_f for each remaining edge $f \notin M$ such that the following holds. The planes $\mathcal{P}$ are chosen to intersect generically. That is, every two planes intersect in a line, every three intersect in a point, and every four are disjoint. Each edge $f \notin M$ is adjacent to exactly two edges e_1, e_2 of the matching M ($e_1, e_2 \in M$), and the point x_f is chosen to be on the line $P_{e_1} \cap P_{e_2}$. Each vertex v is incident to exactly two edges f_1, f_2 not of the matching M ($f_1, f_2 \notin M$), and we let ℓ_v be the line spanning these points $x_{f_1}, x_{f_2} \in \ell_v$. The points $\mathcal{X}$ are chosen so that the lines $\mathcal{L}$ intersect generically. That is, a pair of lines intersects if and only if, they are constrained to be in the same plane or they are constrained to contain the same point for almost every choice of $\mathcal{P}$ and $\mathcal{X}$.

2.4 Proof of Lemma 2

It remains to be seen that the constraints imposed on $\mathcal{L}$ by the above construction imply that a pair of lines intersect if and only if they correspond to adjacent vertices of G.

For any two adjacent vertices $\{u, v\} = g \in E(G)$, we have immediately that ℓ_u and ℓ_v intersect. If

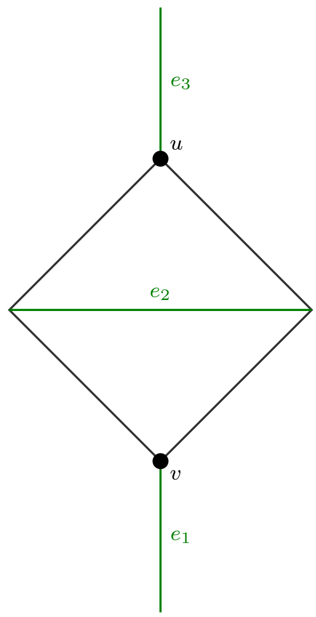

Figure 2: If $e_1, e_2, e_2 \in M$, then ℓ_u and ℓ_v would intersect.

$g \in M$, then we have $\ell_u, \ell_v \subset P_g$, so generically they intersect. Otherwise, $g \notin M$ implies $x_g \in \ell_u, \ell_v$.

For the other direction, consider an edge $e \in M$ and a vertex v that is not incident to e. We claim that the corresponding line ℓ_v is not contained in the corresponding plane P_e. Suppose this were not the case. Let $f_1, f_2 \in E(G) \setminus M$ be the two edges incident to v not of the matching M. Now x_{f_1} and x_{f_2} would both generically be in P_e, which implies f_1 and f_2 are both adjacent to e. But this is impossible, since G is triangle-free. Hence, the claim holds.

To see how the lines $\mathcal{L}$ generically intersect, fix all objects among $\mathcal{P}$ and $\mathcal{X}$ except one point $x \in \mathcal{X}$, which is allowed to vary. Consider a line $\ell \in \mathcal{L}$ depending on x, and let $x' \in \mathcal{X}$ be the other point that ℓ depends on. For any line $\ell' \in \mathcal{L}$ that is not in the same plane as ℓ and does not depend on either x or x', there is only one choice of x that makes ℓ intersect ℓ'. Hence generically, ℓ and ℓ' are disjoint, and likewise any pair of lines in $\mathcal{L}$ corresponding to non-adjacent vertices of G is disjoint.

2.5 Reduction from the Maximum Independent Set Problem

Now, we will construct a polytope K and a collection of segments $\mathcal{S}$ such that each segment is uniquely embedded in K by translation and G is the intersection graph of $\mathcal{S}$. If we can solve MaxSegPack3 for convex polytopes and wish to solve MaxIndSet for a given graph G, we first construct such a pair $(K, \mathcal{S})$, then a maximum packing of segments in K will correspond to a maximum independent set of vertices of G.

Let $\mathcal{L}$ be the collection of lines generated from G

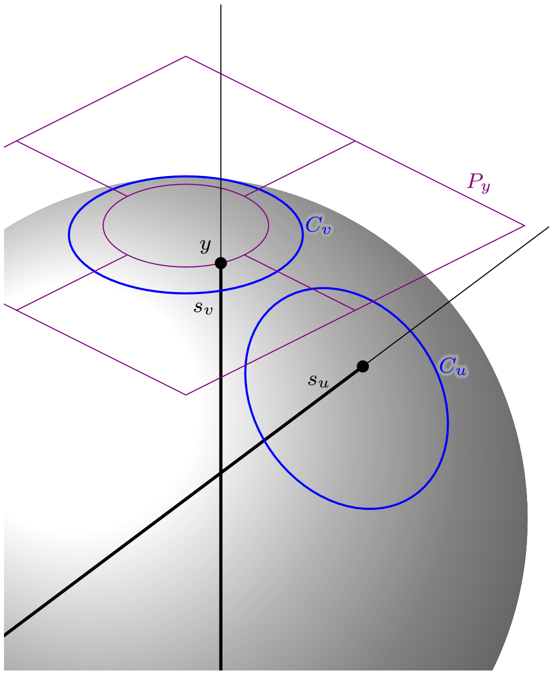

Figure 3: Choosing a sufficiently large sphere ensures that the vertices of each segment in $\mathcal{S}$ will be extreem points of K in the direction of that segment.

as in Section 2.3, and let $r > 0$ be the maximum distance from the origin to any line in $\mathcal{L}$. For each $v \in V(G)$, let C_v be the solid infinite cylinder with axis parallel to ℓ_v passing through the origin and with radius r. Let B a closed ball and S be the boundary sphere of B, such that B centered at the origin with radius sufficiently large so that there is no triple intersection, $C_u \cap C_v \cap S = \emptyset$ for all $u, v \in V(G)$. Let $\mathcal{S} = \{\ell \cap S : \ell \in \mathcal{L}\}$ and $K = \text{conv}(\bigcup \mathcal{S})$. See Figure 3.

2.6 Proof of Theorem 1

To prove that the reduction is correct, we must show two things. First, the pair $(K, \mathcal{S})$ can be constructed in polynomial time from the graph G. Second, the pair does indeed have the properties claimed in Section 2.5.

First we must find a perfect matching $M \subset E(G)$, and this can be done in polynomial time by Edmonds's matching algorithm [4].

Next we must generate a collection of planes that intersect generically. For this, we may choose planes defined by linear equations with coefficients on the moment curve, $P_i = \{p : \langle (i, i^2, i^3), p \rangle = 1\}$ for $i = 1, 2, \ldots, |M|$.

We can then find a polynomial number of points on the intersection of two planes defined in polynomial time, and this is enough to guarantee some generic choice for $\mathcal{X}$, which also gives us $\mathcal{L}$.

Then to compute S, we must compute the maximum distance from the origin to each line, and the minimum angle between pairs of lines. Finally, we can compute the pair $(K, \mathcal{S})$, and this can all be done in polynomial time.

Now, we will show the claimed properties of the construction; that is, (1) each segment in $\mathcal{S}$ is uniquely embedded in K by translation, and (2) G is still an intersection graph of $\mathcal{S}$.

For (1), consider a segment $s_v \in \mathcal{S}$ corresponding to a line $l_v \in \mathcal{L}$ and a vertex y of s_v. Let P_y be the plane that is perpendicular to l_v and contains the vertex y. We claim P_y is a supporting plane of K and $P_y \cap K = y$.

Let H be the closed half-space bounded by P_y that does not contain the center of S. Observe that $H \cap s_y = y$. Any point in $H \cap S$ is at most as far from the axis of C_v as the point y. Therefore, $H \cap S \subset C_v$. By construction, a vertex of any other segment s_u is in $C_u \cap S$. The condition $C_u \cap C_v \cap S = \emptyset$ implies that the set $C_u \cap S$ is disjoint from C_v, and therefore $C_u \cap S$ is disjoint from $H \cap S$ as well, so a vertex of s_u cannot be not in H. Hence, s_u is disjoint from H. Excluding the point y, the segments in $\mathcal{S}$ are entirely on one side of P_y, so P_y is a supporting plane of K and $P_y \cap K = y$.

Now consider a translate t of s_v that is in K. The segment s_v has two such supporting planes and t must be in the space between this parallel pair of supporting planes, since K is between these planes. Therefore, t must intersect each plane. Furthermore, t must intersect each supporting plane at a point in K. Since these supporting planes only intersect K at a single point each, there can be only one possible translate, namely $t = s_v$ itself.

Remark. In particular, for a segment $s_v \in \mathcal{S}$ with end points y, z, and for any other point $x \in K$, x is away from the plane P_y, which is perpendicular to $s_v = yz$, and on the same side of P_y as z. This implies that the angle between xy and yz is acute. For s_v to translate in K it would have to translate away from P_y, while similarly translating away from P_z as well.

For (2), we only need to show all the intersections of pairs of lines are inside K, since the corresponding segments are uniquely embedded. Let l_v, l_u be an intersecting pair of lines and C_v, C_u be the corresponding cylinders defined as in Section 2.5. Since $C_v \cap C_u \cap S = \emptyset$ and $C_v \cap C_u$ is convex and contains the center of S, $C_v \cap C_u$ is contained in B. Consequently, $l_v \cap l_u \subset C_v \cap C_u$ is also contained in B. In particular, $l_v \cap l_u$ is in $s_v = l_v \cap B$, which is in K. Therefore, $l_v \cap l_u$ is inside K.

Finally, since the sets of segments among $\mathcal{S}$ that can be packed in K correspond to the independent sets of G, MaxSegPack3 for convex polytopes is at least within a polynomial factor as hard as MaxIndSet for bridgeless triangle-free cubic graphs. Thus, MaxSegPack3 for convex polytopes is NP-hard.

2.7 Case II

Theorem 3 MaxSegPack2 *for general polygonal regions is NP-hard*

Proof. Again this is by reduction to MaxIndSet. MaxIndSet restricted to planar graphs in addition to all the restrictions in the proof of Theorem 1 is still NP-hard [11]. Any triangle-free planar graph can be realized in polynomial time as the intersection graph of segments $\mathcal{S}$ in the plane [3]. The reduction then follows by packing the maximum number of segments among $\mathcal{S}$ in the region $\bigcup \mathcal{S}$. □

3 Conclusion

Many related questions remain open. Theorem 1 only provides a lower bound on the complexity of MaxSegPack3 for convex polytopes. It remains to provide an upper bound on complexity. It also remains to provide an approximation algorithm, through even a constant factor approximation algorithm may be too much to hope for [2]. MaxIndSet is in general also known to be $W[1]$-hard, but $W[1]$-hardness for this problem also still remains open because we used only bridgeless triangle-free cubic graphs for the reduction.

This can all be said for MaxSegPack2 for general polygonal regions as well, and we do not yet know of any notable bounds on the complexity of MaxSegPack2 for convex polygonal regions.

We may also consider the decision version of these packing problems. Given a collection of segments, a region in $\mathbb{R}^d$, and a value n, can at least n of the segments be disjointly embedded in the given region. While this problem has lower complexity than MaxSegPack$d$, we know from Theorem 1 that there cannot be a polynomial time algorithm for convex polytopes, and likewise by Theorem 3 for general polygonal regions.

The minimum segment-packaging problem that we saw in the introduction also remains open, both to determine its complexity and to provide an efficient algorithm. Approximation algorithms would also be of interest here, especially if the minimum-packaging problem is intractable. In an unpublished communication Otfried Cheong has given a polynomial time 3-approximation algorithm.

We may also consider the corresponding decision problem. Given a collection of segments and values m and n, is there a convex set K of measure at most m such that at least n of the segments can be disjointly embedded in K. This is closely related to the packing problems considered here. The decision version of the maximum-packing problem coincides with the decision version of the minimum-packaging problem for a specific region of a given area.

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